1 or 0? Cantorian conundrums in the contemporary classroom

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Introduction

In set theory, one comes across the notion of "vacuous truth". A statement is vacuously true if it is true but does not quite say anything. The structure of a vacuously true statement is typically of the form: everything with property A also has property B, with the caveat being that there is nothing in property A. For instance we could say: all humans with gills are sharks. This statement is vacuously true because there are no humans with gills. It is natural to dismiss such examples as absurd and pathologies within the framework of set theory. However the notion of vacuous truth arises in some pedagogical situations. The reader is undoubtedly curious whether a situation requiring the examination of "vacuous" truth can arise in a contemporary mathematics classroom. In fact such situations do arise! One such situation in a pre-service elementary mathematics classroom will be described, discussed and analysed for classroom implications. Cantor's seminal work, which showed infinity comes in various sizes(!) was instrumental in the development of modern set theory. Cantor's contributions can be viewed as the "ur-source"[1] of the consequent paradoxes that arose within set theory. In his honour, we will label unusual set-theoretic pedagogical situations *Cantorian conundrums.*

The situation

In the United States, prospective elementary school teachers are required to take some mathematics content courses at the university level which typically cover some elementary set theory to act as a foundational base and a context out of which models for the four arithmetic operations (+, − , ×, ÷) are developed. The situation we are about to describe took place in one such course. The following problem was part of the homework assigned to the students. At

1. In German and Swedish the syllable and preposition *ur* means "something original" or "the source"

this stage of the course students were familiar with set theoretic notions of subsets, unions, intersections, complements etc.

> Suppose B is a proper subset of C. If n(C) = 8, what is the maximum number of elements in B? What is the least possible number of elements in B? (Billstein, Libeskind & Lott, 2004, p. 74).

At first glance the problem appears as a rudimentary set theory problem. However the ensuing discussion of this problem in the classroom revealed otherwise. The entire class agreed that the answer to the first part of the question was 7. In order to justify this, students appealed to the definition of a proper subset and discarding the one subset, namely the set itself. So far so good! When we got to the second part of the problem, a show of hands revealed that the class was divided 12–9 about the answer. Twelve students thought the answer was 0, whereas nine students claimed that the answer was 1. This was an interesting turn of events and the ideal moment to step back and ask students in either camp to justify their claims. The identity of the students in the respective camps was also noted. The more vocal of the students in the "zero camp" essentially argued that one simply lists out all the proper subsets of the given set, and observes that the empty set is one of these proper subsets. Since there are no elements in the empty set, 0 is the least number of elements that B can have. On the other hand, students in the "one camp" argued, "If the answer is zero, then aren't we viewing the empty set as an element? We cannot view the empty set as an element, so the answer should be one" (Class comment). Towards the end of the class period, the class was still far from reaching a consensus. This was the ideal opportunity to let students in either camp reflect on their choice. The "positivist" turn of events was too good a pedagogic opportunity to pass up. Hence following written assignment was announced.

Taking and defending a position

Choose a position (1 or 0). Based on your position, write a two-paragraph "opinion" defending the validity of your choice.

The written opinions, collected three days later, revealed a 14–6–1 split. Fourteen now favoured 0, six of the nine original students still favoured 1 (no switches), and one student from the one-camp had shifted to "an answer dependent on vacuous or logical truth" favouring zero. Some sample responses from the three camps are now presented. These will help illuminate the subsequent discussion and implications sections of the paper.

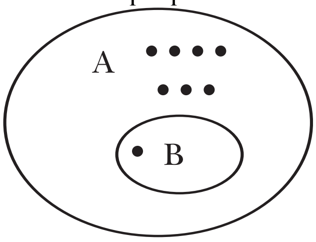
The "one camp"

> My position supports the "1 side"… The question is: is the empty set an element? No it is not. The definition (of proper subsets) states: "every element

of B is contained in A and there is at least one element of A that is not in B". However this definition does not state anything about the empty set… the question asks for the number of elements, an empty set is not an element. Therefore the answer is 1.

The empty set is a set with no elements. The empty set is not an element in a set but a subset or a set in general… Since the empty set is not an element but a set with no elements it cannot be considered (for) the least number of elements.

I am guessing… that the answer is 1. I would have to say this because you have to have one element from C to be a proper subset at all. See figure:



Emptiness cannot be viewed as a common property between sets.

The definition of an element is: individual objects in a set. Although the empty set is contained in each set, it is not one of the objects we take into account when viewing the number of objects. Since B is a subset of C, it has to always have at least one object in common with the original set C. The following example conveys this idea: Set C = {a,b,c,d,e,f}, Set B = {a}. In both these sets the empty set does not appear directly in the set.

Suppose we start with the empty set $\varnothing$. How many elements does a subset of this set have? It is the set itself! So the answer is 1.

The "zero camp"

The set in question has a cardinal number 8, meaning there are 8 elements in set C. Let's say C = {1,2,3,4,5,6,7,8}, and we are going to list all the subsets of set C. The subsets of C are { }, {1}, {2},… I am not going to keep going because the question is already answered. The empty set is a subset of C and { } contains zero elements. Therefore zero is the correct answer.

I believe the answer to this question can be best described in a story… Pretend you have a bag of marbles. The number of marbles in this instant doesn't matter, what matters is that inside the bag… is a group of elements. These can be grouped even if you make a group of 0 elements. So as told in this marble story, you can have 0 elements because everything that is grouped inside the bag is still considered an element.

Here is an example to prove that zero is the correct answer. If a set contains

only one element, f, then its subsets include the empty set, which has zero elements, and {f}, which is only one element. The only proper subset in this situation is the empty set. Therefore zero is the least number of elements.

The camp between camps

The sum of the empty set is 0 (the nullary sum). Therefore the cardinality of the empty set is 0. For any set, in this case C, the empty set is a subset of C. Hence B, as a proper subset of C, could be the empty set. For example, if a set is a bag of elements, an empty bag may be empty (0 elements) but it still exists! There also exists a loophole called vacuous truth: a truth while "true" doesn't say anything. This stated, then, a set is finite if its cardinality is a natural number. But 0 is not a natural number. Therefore our vacuous truth is: the cardinality of the empty set is 0, the empty set is finite…So while the empty set itself contains no elements to speak of, in and of itself, it still exists as a subset. So logically speaking the answer is 1, and vacuously speaking the answer is 0.

Discussion and implications

As the written responses reveal, the problem was a source of considerable contention, introspection, and led students to a careful re-examination of the definitions under use. If the goal of teacher training is to produce teachers that memorise and regurgitate mathematics, such an exercise could be construed as a waste of time. For those that value the final "positivist" black and white product, the answer is the end-all. On the other hand if the goal of teacher training is to create teachers who value the role of argumentation in the making of mathematics; i.e., the process over the product, such an exercise is invaluable towards pursuing this pedagogic goal. Although there were flaws in the reasoning of the "one camp", such as considering the empty set as an element of every set as opposed to a subset, many of these students stumbled upon Cantorian conundrums in this process. Questions such as, "Can emptiness be viewed as a common property between sets?" and, "How many elements does the subset of an empty set have?" are more philosophical in nature, than they are mathematical! The underlying affective hope here is that prospective teachers' experiences of an university educator valuing their thoughts will translate into them as teachers valuing young children's thoughts. The philosophical questions posed by the students are also an avenue through which the university educator can initiate an exploration of related set theoretic topics via projects. For instance the question "What are the subsets of the empty set?" leads into a better grasp of the notion of power sets. Since every set has the empty set as a subset, the empty set is one of those very sets. Therefore it has itself as a subset. Which means having itself as a subset leads to itself as a member of its power set! In conclusion, we are reminded of Mason's (2003) recent exposition on the structure of attention from a cognitive standpoint. Although the context here is more argumenta-

tive, Mason's framework is certainly applicable from the observer's standpoint. In students' responses, one is offered an insight into "the lived experience of mathematical thinking", and observes to an extent the structure of their attention on abstract notions such as, elements of a set, subsets and the empty set. One conjectures that there is a change in the structure of attention about elements of a subset as evidenced in the number of students moving from the one-camp to the zero-camp! As an added bonus, the student in the "camp between camps" truly experienced the legacy of Cantorian conundrums!

References

Billstein, R., Libeskind, S. & Lott, J (2004). *A Problem Solving Approach to Mathematics for Elementary School Teachers (8th Ed.).* Addison Wesley Longman.

Mason, J. (2003) .On the structure of attention in the learning of mathematics. *The Australian Mathematics Teacher, 59*(4), 17–25.